A new form of self-duality equations with topological term

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Based on the $U(1)$ gauge potential decomposition theory and ϕ-mapping theory, the topological inner structure of the self-duality (Bogomol'nyi-type) equations are studied. The special form of the gauge potential decomposition is obtained directly from the first of the self-duality equations. Using this decomposition, the topological inner structure of the Chern-Simons-Higgs (CSH) vortex is discussed. Furthermore, we obtain a rigorous self-dual equation with topological term for the first time, in which the topological term has been ignored by other physicists.

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I. INTRODUCTION

Bogomol'nyi-type vortices and self-dual solutions appear in a variety of gauge theories scenarios in $2+1$ dimension, including Abrikosov-Nielson-Olesen model [1, 2, 3], Chern-Simons-Higgs (CSH) Model [4, 5, 6], Maxwell-Chern-Simons Higgs Model [7] and Jackiw-Pi model [8].

Let us start with an Abelian CSH Lagrangian density [4]

$$L_{CSH} = \frac{1}{4}\kappa\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} + D_\mu\phi(D_\mu\phi)^* - V(\|\phi\|), \tag{1}$$

where ϕ is the designated charged Higgs complex scalar field minimally coupled to an Abelian gauge field, $\frac{1}{4}\kappa\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}$ is the so-called Chern-Simons term, and the covariant derivative is $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$. The first-order Bogomol'nyi

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self-duality equations in this model are

$$D_\pm\phi = 0 \quad (D_\pm \equiv D_1 \pm iD_2),$$
$$B = \pm\frac{2}{\kappa^2}\|\phi\|^2(\|\phi\|^2 - \nu^2). \tag{2}$$

As pointed out by many physicists [4, 9], the magnetic flux of the vortex is

$$\Phi = \oint A_i dx^i = \int \epsilon^{ij}\partial_i A_j dx^2 = \frac{2\pi n}{e}, \tag{3}$$

here n is a topological index characterizing the vortex configuration. When the scalar field ϕ is decomposed into its phase and magnitude: $\phi = \rho^{\frac{1}{2}}e^{iw}$, the first of the self-duality equations (2) determines the gauge field:

$$A_i = -\frac{1}{e}\partial_i w \mp \frac{1}{2e}\epsilon_{ij}\partial_j \ln\rho. \tag{4}$$

The second of Eq. (2) reduces to a nonlinear elliptic equation for the scalar field density ρ:

$$\bigtriangledown^2 \ln\rho = \frac{4e}{\kappa^2}\rho(\rho - \nu^2), \tag{5}$$

and this equation is not solvable, or even integrable.

In this paper we accomplish two things. Firstly, using Duan's ϕ-mapping theory [10, 11, 12, 13, 14], we study the topological inner structure of the first Bogomol'nyi self-duality equation and obtain directly one special form of the gauge potential decomposition. Using this decomposition, we derive the vortex configuration given in Eq. (3) from the self-duality equations (2), and one sees that the inner structure of this vortex labelled only by the topological indices of the zero points of the complex scalar field. Secondly, we study the second self-duality equation in Eqs. (2) and obtain a new scalar field equation with a topological term, which differs from the conventional equation (5), and the topological term is the topological current of the vortex. Furthermore, the analytical solution of the equation in the special conditions is also investigated.

II. U(1) GAUGE POTENTIAL DECOMPOSITION OF SELF-DUALITY EQUATIONS

It is well known that the complex scalar field ϕ can be looked upon as a section of a complex line bundle with base manifold M [15]. Denoting the charged Higgs complex scalar field ϕ as

$$\phi = \phi^1 + i\phi^2, \tag{6}$$

where $\phi^a(a = 1, 2)$ are two components of a two-dimensional vector field $\vec{\phi} = (\phi^1, \phi^2)$ over the base space, one can introduce the two-dimensional unit vector

$$n^a = \frac{\phi^a}{\|\phi\|}, \quad \|\phi\| = (\phi\phi^*)^{\frac{1}{2}}. \tag{7}$$

Let us consider the first of self-duality equations (2) (firstly, we choose the upper signs):

$$D_+\phi = 0, \tag{8}$$

substituting Eq. (6) into the above equation, we obtain two equations

$$\begin{aligned} \partial_1\phi^1 - \partial_2\phi^2 &= eA_1\phi^2 + eA_2\phi^1, \\ \partial_1\phi^2 + \partial_2\phi^1 &= eA_2\phi^2 - eA_1\phi^1. \end{aligned} \tag{9}$$

Making use of the above relations, we derive:

$$\partial_1\phi^*\phi - \partial_1\phi\phi^* = 2ieA_1\|\phi\|^2 + i(\partial_2\phi^*\phi + \partial_2\phi\phi^*), \tag{10}$$

$$\partial_2\phi^*\phi - \partial_2\phi\phi^* = 2ieA_2\|\phi\|^2 - i(\partial_1\phi^*\phi + \partial_1\phi\phi^*). \tag{11}$$

To proceed, we need a fundamental identity-one that appear many times throughout our study in the gauge potential decomposition theory:

$$\epsilon_{ab}n^a\partial_i n^b = \frac{1}{2i}\frac{1}{\phi^*\phi}(\partial_i\phi^*\phi - \partial_i\phi\phi^*), \tag{12}$$

using this identity, Eq. (10) and Eq. (11) become

$$eA_1 = \epsilon_{ab}n^a\partial_1 n^b - \frac{1}{2}\partial_2\ln(\phi\phi^*), \tag{13}$$

$$eA_2 = \epsilon_{ab}n^a\partial_2 n^b + \frac{1}{2}\partial_1\ln(\phi\phi^*). \tag{14}$$

Eqs. (13) and (14) can be rewritten as:

$$eA_i = \epsilon_{ab}n^a\partial_i n^b - \epsilon_{ij}\frac{1}{2}\partial_j\ln(\phi\phi^*). \tag{15}$$

Following the same discussion, we obtain the similar equation from $D_-\phi = 0$:

$$eA_i = \epsilon_{ab}n^a\partial_i n^b + \epsilon_{ij}\frac{1}{2}\partial_j\ln(\phi\phi^*). \tag{16}$$

So, from the first self-duality equation $D_\pm\phi = 0$, we get

$$A_i = \frac{1}{e}\epsilon_{ab}n^a\partial_i n^b \mp \frac{1}{2e}\epsilon_{ij}\partial_j \ln(\phi\phi^*). \tag{17}$$

The $U(1)$ gauge potential can be decomposed by the Higgs complex scalar field ϕ as

$$A_i = \beta\epsilon_{ab}\partial_i n^a n^b + \partial_i\lambda, \tag{18}$$

in which $\beta = \frac{1}{e}$ is a constant and $\lambda = \mp\frac{1}{2e}\epsilon_{ij}\ln(\phi\phi^*)$ is a phase factor denoting the $U(1)$ transformation. It is seen that the term $(\partial_i\lambda)$ in Eq. (18) contributes nothing to the magnetic flux of the vortex (3).

From above discussion, it is obvious to see that we obtain one special form of the general $U(1)$ gauge potential decomposition from the first self-duality equation, in the next section, we will see that the topological inner structure of CSH vortex is described by gauge potential A_i (17).

III. THE TOPOLOGICAL INNER STRUCTURE OF CSH VORTEX

Based on the decomposition of the gauge potential A_i discussed in section II, we can immediately get the equation of the magnetic flux of the vortex (3). Using the two-dimensional unit vector field (7), we can construct a topological current:

$$J^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda = \frac{1}{e}\epsilon^{\mu\nu\lambda}\epsilon_{ab}\partial_\nu n^a\partial_\lambda n^b, \tag{19}$$

which is the special case of the general ϕ-mapping topological current theory [10]. Obviously, the current (19) is conserved. Following the ϕ-mapping theory, it can be rigorously proved that

$$J^\mu = \frac{2\pi}{e}\delta^2(\vec{\phi})D^\mu(\frac{\phi}{x}), \tag{20}$$

where $D^\mu(\frac{\phi}{x}) = \frac{1}{2}\epsilon^{\mu\nu\lambda}\epsilon_{ab}\partial_\nu\phi^a\partial_\lambda\phi^b$ is the vector Jacobians. This expression provides an important conclusion: $J^\mu = 0,\ iff\ \vec{\phi} \neq 0;\ J^\mu \neq 0,\ iff\ \vec{\phi} = 0$. Suppose that the vector field $\vec{\phi}(\phi^1, \phi^2)$ possesses l zeros, denoted as $z_i (i = 1, ..., l)$. According to the implicit function theorem [16], when the zero points $\vec{z_i}$ are the regular points of $\vec{\phi}$, that requires the Jacobians determinant

$$D\left(\frac{\phi}{x}\right)\Big|_{z_i} \equiv D^0\left(\frac{\phi}{x}\right)\Big|_{z_i} \neq 0. \tag{21}$$

The solutions of Eq. (21) can be generally obtained: $\vec{x} = \vec{z_i}(t),\ i = 1, 2, \cdots, l,\ x^0 = t$. Using Eqs. (19) and (21), it is easy to prove that

$$D^\mu\left(\frac{\phi}{x}\right)\Big|_{z_i} = D\left(\frac{\phi}{x}\right)\Big|_{z_i}\frac{dx^\mu}{dt}. \tag{22}$$

According to the δ-function theory [17] and the ϕ-mapping theory, one can prove that

$$J^{\mu} = \frac{2\pi}{e}\sum_{i=1}^{l}\beta_i\eta_i\delta^2(\vec{x}-\vec{z}_i)\frac{dx^{\mu}}{dt}\Big|_{z_i}, \tag{23}$$

in which the positive integer β_i is the Hopf index and $\eta_i = sgn(D(\phi/x)_{z_i}) = \pm 1$ is the Brouwer degree [11, 18]. Then the density of topological charge can be expressed as

$$J^0 = \frac{2\pi}{e}\sum_{i=1}^{l}\beta_i\eta_i\delta^2(\vec{x}-\vec{z}_i). \tag{24}$$

From Eq. (19), it is easy to see that

$$J^0 = \epsilon^{ij}\partial_i A_j. \tag{25}$$

So, the total charge of the system given in Eq. (3) can be rewritten as

$$Q = \int J^0 dx^2 = \frac{2\pi}{e}\sum_{i=1}^{l}\beta_i\eta_i. \tag{26}$$

And the topological index n in Eq. (3) has the following expression

$$n = \sum_{i=1}^{l}\beta_i\eta_i. \tag{27}$$

It is obvious to see that there exist l isolated vortices in which the ith vortex possesses charge $\frac{2\pi}{e}\beta_i\eta_i$. The vortex corresponds to $\eta_i = +1$, while the antivortex corresponds to $\eta_i = -1$. One can conclude that vortex configuration given in Eq. (3) is a multivortex solution which possesses the inner structure described by expression (26).

IV. SELF-DUAL EQUATION WITH TOPOLOGICAL TERM

The second Bogomol'nyi self-duality equation (5) is meaningless, when the field $\phi = 0$. Moreover, no exact solutions are known for this equation. In this section, based on the decomposition of $U(1)$, we rewrite the self-dual equation with topological term and study its general analytical solution in some condition.

Firstly, based on the special form of the general $U(1)$ decomposition of gauge potential (17), we get:

$$B = \frac{1}{e}\epsilon^{ij}\epsilon_{ab}\partial_i n^a\partial_j n^b \mp \frac{1}{2e}\delta_{jk}\epsilon^{ij}\epsilon^{kl}\partial_i\partial_l \ln(\phi\phi^*). \tag{28}$$

Substituting above equation into the second Bogomol'nyi self-duality equation (2), we obtain:

$$\frac{1}{2}\bigtriangledown^2 \ln(\phi\phi^*) = \frac{2e}{\kappa^2}\|\phi\|^2(\|\phi\|^2-\nu^2) \mp \epsilon^{ij}\epsilon_{ab}\partial_i n^a\partial_j n^b. \tag{29}$$

From Eqs. (19) and (20), we obtain the δ-function form of topological term

$$\epsilon^{ij}\epsilon_{ab}\partial_i n^a \partial_j n^b = eJ^0 = 2\pi\delta^2(\vec{\phi})D(\frac{\phi}{x}). \tag{30}$$

The second self-dual equation in Eq. (2) then can reduce to a nonlinear elliptic equation for the scalar field density ($\rho = \phi\phi^*$)

$$\bigtriangledown^2 \ln\rho = \frac{4e}{\kappa^2}\rho(\rho-\nu^2) \mp 4\pi\delta^2(\vec{\phi})D(\frac{\phi}{x}). \tag{31}$$

Comparing with Eq.(5), one can find that the conventional self-dual equation (5), in which the topological term has been ignored, is meaningless when the field $\phi = 0$; we get the self-dual equation with topological term, which is meaning when the field $\phi = 0$. Obviously, topological term is very important to the inner topological structure of the self-duality equations, and $\mp 4\pi\delta^2(\vec{\phi})D(\frac{\phi}{x})$ is the density of topological charge of the vortex.

Now the self-dual equation with topological term is more difficult to be solved, we will study the analytical solution of the equation in the following condition:

$$\frac{4e}{\kappa^2}\rho\nu^2 \pm 4\pi\delta^2(\vec{\phi})D(\frac{\phi}{x}) = 0. \tag{32}$$

In this case, Eq. (31) reads as

$$\bigtriangledown^2 \ln\rho = \frac{4e}{\kappa^2}\rho^2. \tag{33}$$

Let $\rho = \sqrt{f}$, then Eq. (33) gives

$$\bigtriangledown^2 \ln f = \frac{8e}{\kappa^2}f, \tag{34}$$

which is a nonlinear equation known as the Liouville equation, and has the general exact solutions

$$f = \frac{\kappa^2}{4e}\nabla^2\ln(1+|g|^2), \tag{35}$$

where $g = g(z)$ is a holomorphic function of $z = x^1 + ix^2$. So we obtain the general analytical solution of Eq. (33):

$$\rho = \sqrt{\frac{\kappa^2}{4e}\nabla^2\ln(1+|g|^2)}\,, \tag{36}$$

where the corresponding charge density ρ of the vortex must obey the condition (32), so the holomorphic function $g = g(z)$ is not arbitrary.

V. CONCLUSION

Our investigation is based on the connection between the self-dual equation of Chern-Simons-Higgs model and the $U(1)$ gauge potential decomposition theory and ϕ-mapping theory. First, we directly obtain one special form of $U(1)$ gauge potential decomposition from the first of the self-duality equations. Moreover, we obtain the inner topological structure of the Chern-Simons vortex. The multicharged vortex has been found at the Jacbian determinate $D(\phi/x) \neq 0$. It is also showed that the charge of the vortex is determined by Hopf indices and Brouwer degrees. Second, we establish the rigorous self-duality equations with topological term for the first time, in which the topological term is the density of topological charge of vortex: $4\pi\delta^2(\vec{\phi})D(\frac{\phi}{x}) = 2eJ^0$. In contrast with the conventional self-duality equation (5), one can see that the self-duality equation with topological term is valid when the field $\phi = 0$; topological term vanishes and the self-duality equation becomes Eq. (5) when the field $\phi \neq 0$. Additionally, the analytical vortex solution of the equations in the special condition (32) is obtained, the charge density ρ of the vortex must obey the condition.

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